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A Study on the Mathematical Modelling of Population Change

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ABSTRACT

Rapidly expanding human and animal populations are frequently described by the exponential function; however, this growth is not permanently sustainable. If only because their resource base will inevitably erode, growing populations must eventually stabilize or even collapse. The question of when and to what extent the global human population will stabilize is hotly debated. The population levels off as the environment's carrying capacity approaches thanks to the widely studied logistic or sigmoidal function and its distinctive S shape. The goal of the multidisciplinary academic area of bio-mathematical modeling is to use applied mathematics approaches to model biological and natural processes. The study of population dynamics is becoming more and more popular in the early twentieth century. The study of population dynamics, which combines the disciplines of mathematics, demography, social sciences, ecology, population genetics, and epidemiology, aims to provide a straightforward, mechanistic explanation of how the size and makeup of biological populations—such as those of humans, animals, plants, or microorganisms—change over time.

INTRODUCTION

Estimating the values of the many parameters in the mathematical functions used to characterize the population trajectory is a common challenge in modeling population dynamics, whether it be in people or animals. Numerous numerical approximation techniques must be employed because, in most cases, neither the population's eventual maximum size nor its rate of increase are known. This article presents a mathematical approach to population projection that circumvents these challenges by starting with the assumption that the populations from just three censuses provide the entire data available for calculation. It is a technique that Keyfitz suggested but did not create [4].

Although there is a carrying capacity that a population cannot surpass, there is no reason why it shouldn't drop after reaching this level, even to zero, which is as implausible as it may be but not impossible. As demonstrated by the population growth and eventual collapse on Easter Island between the eighth and fourteenth centuries AD, there is historical precedent for this (see [5, 6]). The so-called normal distribution function, which is so well-known in statistical analysis, could explain the upward and then downward trend.

Here, the ability of the logistic and normal functions—shown in Figure 1.1—to provide long-term population projections is examined. However, the population projection provided by the application of a "normal function" is not in any way regularly distributed.

The Basic Equations For Population Change

Let a, b, and c, as well as p, q, and r, be constants that need to be found, and let Pt and t be the population and time variables, respectively.

There is an initial interest in defining Pt′ and then solving the resulting differential equation to derive an expression for Pt because population change is inevitably a function of time. Five population models that span the entire spectrum of possibilities—from ultimate extinction to endless growth—are summarized in Table 2.

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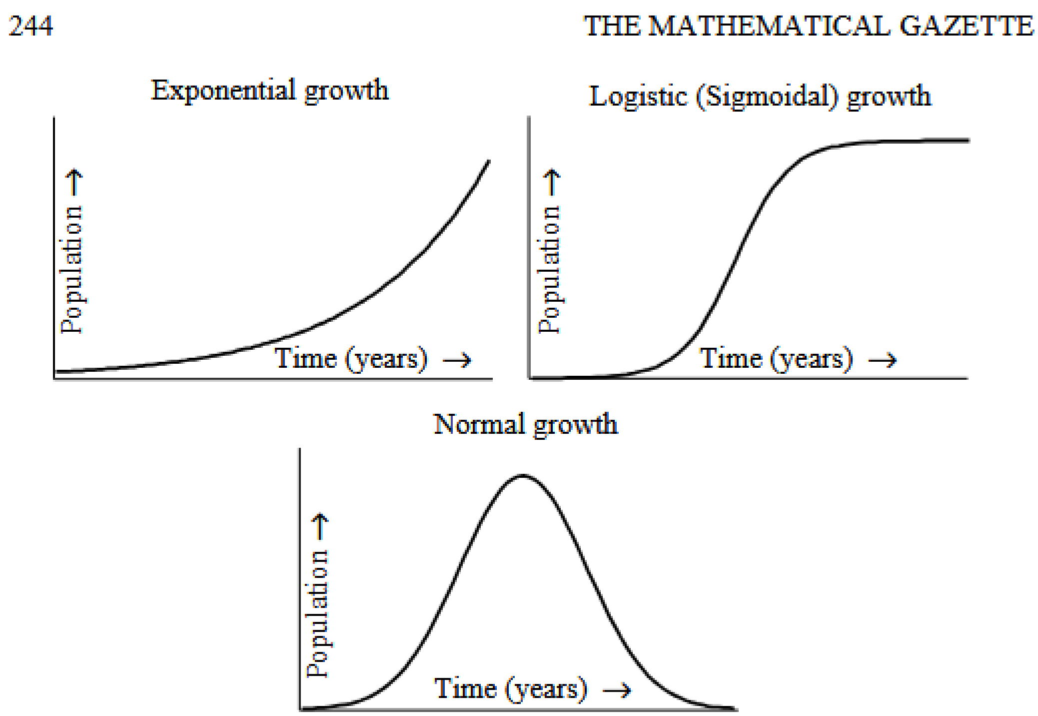

Figure 1: Exponential, Logistic And Normal Growth Functions

Table 2: Mathematical Models of Population Change

Population trend	*Rate of change of population*			*Population model*		
No change; stationary	P_t'	=	0	P_t	=	c
Linear change; unrealistic	P_t'	=	c	P_t	=	$b + ct$
Exponential growth; unsustainable	P_t'	=	cP_t	P_t	=	be^{ct}
Logistic (sigmoidal) growth; ultimate stability	P_t'	=	$b^{P_t}(1 - a)$ $- c - P_t$	P_t	=	$\frac{a}{1 + be^{-ct}}$
Normal function change; growth followed by decline; ultimate extinction	P_t'	=	$(q - 2rt)P_t$	P_t	=	$e^{p + qt - rt^2}$

It is important to note that Table 2 does not include all potential population models. It may be possible to describe growth and stabilization using arctan or hyperbolic functions of the form Pt = a + b arctan ct and Pt = a + b tanh ct, respectively. They are not further studied, though, because the assessment of the c parameter necessitates the series expansion of these functions, which calls for computer-based solutions.

According to Table 2, the linear growth scenario of Pt′ = c and the situation of Pt′ = 0 are both essentially trivial and are therefore not taken into consideration further, if only because they both have minimal impact on the actual population increase. From a modern standpoint, however, population growth was roughly linear with a very low gradient from the earliest ages to about 1800, but not subsequently.

Even though it can be applied to specific time-constrained periods, the limitless growth described by the widely studied exponential function, Pt′ = cPt, also presents an unrealistic long-term growth model. Since there are only two unknown parameters in the exponential growth model, there is no significant mathematical difficulty in its practical application, hence it is not further explored here.

However, in addition to providing believable, albeit not necessarily trustworthy, long-term projections, the logistic and normal functions are more mathematically intriguing due to the three unknown parameters they contain, which makes their practical application more challenging. These two roles are discussed in more detail below.

Logistic (sigmoidal) population growth
From Table 2

$$P_t = \frac{a}{1 + be^{-ct}}. \qquad (1)$$

When $t = 0$ then

$$b = \frac{a}{P_1} - 1 \qquad (2)$$

The population at t = 0 is denoted by P1, and from which

$$P_t = \frac{aP_1}{P_1 + (a - P_1)e^{-ct}}. \qquad (3)$$

Assume that the population at each census is P1, P2, and P3 at dates t1, t2, and t3, with n years separating them, so that 2t2 = t3 + t1. The time variable can be removed by sampling the population at equal intervals (Figure 3). If j < k and Pj and Pk are the populations at censuses j and k, respectively, then (3) indicates that

$$P_k = \frac{aP_j}{P_j + (a - P_j)e^{-c(k-j)n}}$$

Thus, after rearranging to remove the time variable, yields j = 1, k = 2, and j = 1, k = 3 in that order.

$$a = \frac{2P_1P_2P_3 - P_2^2(P_1 + P_3)}{P_1P_3 - P_2^2}. \qquad (4)$$

Putting (4) into (2) and then using (3), gives

$$b = \frac{P_3(P_1 - P_2)^2}{P_1(P_2^2 - P_1P_3)} \quad \text{and} \quad c = \frac{1}{n}\ln\left(\frac{P_3(P_2 - P_1)}{P_1(P_3 - P_2)}\right).$$

Since values for a, b, and c have now been discovered, (1) can be used to calculate any population size. It is necessary that t = T − t1 in (1) if T is the year of interest.

Since Pt = a (Table 2) or 0 in the event of the minimal population, let a = Pmax, the maximum population reached.

Since the logistic function is asymptotic to its maximum (and minimum) values, the maximum population is effectively never reached when t → ∞, as shown in (1).

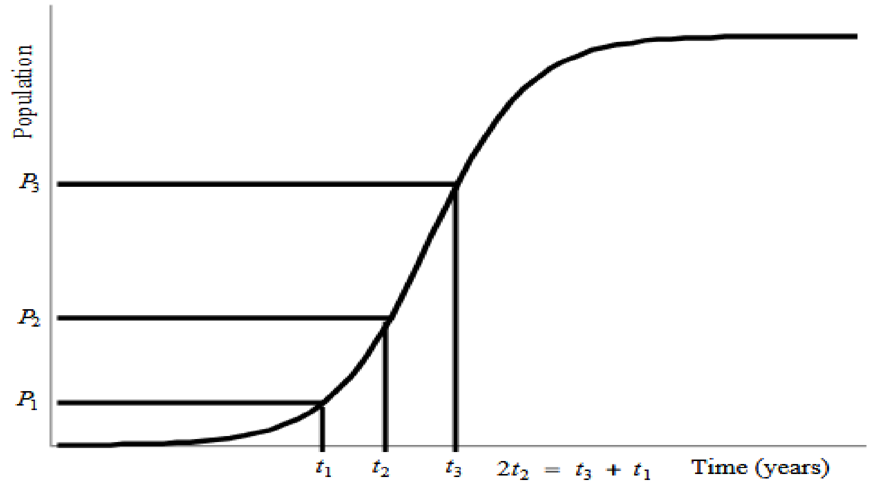

Figure 3: Methodology used in modelling logistic population growth

Keep in mind that the model only works if P2 > P P because

$$P_{\max} > P_3 \Rightarrow P_{\max} - P_3 > 0$$

and hence from (4)

$$\frac{-P_1(P_2 - P_3)^2}{P_1P_3 - P_2^2} > 0;$$

but

$$(P_2 - P_3)^2 > 0$$

hence

$$P_2^2 > P_1P_3. \qquad (5)$$

It should be noted that in the limiting case, when P2 = P1P3, (5) implies that Pmax → ∞ as 2 → P1P3, and that the function eventually simplifies to one of exponential growth.

Normal Function Population Growth

From Table 2

$$P_t = e^{(p + qt - rt^2)}. \qquad (6)$$

P1 = ep when t = 0, where P1 is the starting population.
p = ln P1
and (6) becomes into

$$P_t = P_1 e^{(qt - rt^2)}$$

function that increases monotonically.

Using P1, P2, …, Pk with n years between each, and then between censuses j and k with j < k, we proceed as in the previous section.

$$P_k = P_j e^{\{q(k-j)n - r(k-j)^2 n^2\}}.$$

Once more, with only three censuses in the minimalist scenario, j = 1 and k = 2, 3, then by rearrangement

$$q = \frac{1}{2n}\left\{\ln\left(\frac{P_2^4}{P_1^3 P_3}\right)\right\} \qquad (7)$$

and

$$r = \frac{1}{2n^2}\left\{\ln\left(\frac{P_2^2}{P_1 P_3}\right)\right\}. \qquad (8)$$

With t = T − t1, values for p, q, and r have been discovered that allow for the determination of any population size using (6).

Since r > 0 is the model's validity constraint, (8) implies that P2 > P1P3, which is equivalent to (5); thus, the same constraint holds true for both logistic and normal growth functions. If 2 < P1P2, then r < 0 and exponential growth is still true according to (6).

When Pt′ = 0 (Table 2), the maximum population, Pmax, occurs; as a result, (7) and (8) are rearranged.

$$t = \frac{q}{2r} = \frac{n(3\ln P_1 - 4\ln P_2 + \ln P_3)}{2(\ln P_1 - 2\ln P_2 + \ln P_3)}. \qquad (9)$$

When t is arbitrarily big (t → ∞) and the population has gone extinct, the minimum population occurs because r > 0. With P0 = P1, substituting (9) into (6) yields the maximum population reached.

$$P_{\max} = P_1 e^{\frac{q^2}{4r}} = P_1 e^{\frac{3\ln P_1 - 4\ln P_2 + \ln P_3}{8(\ln P_1 - 2\ln P_2 + \ln P_3)}}. \qquad (10)$$

Now that every unknown parameter in the logistic and normal functions has been identified, a real-world example is provided below.

APPLICATION TO GLOBAL POPULATION GROWTH

Table 4 gives UN estimates of world population between 1950 and 2010 at five year intervals.

1950	2.53		1975	4.08		2000	6.12
1955	2.77		1980	4.45		2005	6.51
1960	3.04		1985	4.86		2010	6.90
1965	3.33		1990	5.31			
1970	3.70		1995	5.73			

Table 4: UN global population estimates (billions) 1950-2010
Data Source: UN Population Division [7]

Data for 1950, 1980, and 2010 (2.53, 4.45, and 6.90 billions respectively) are the best choices since they employ the longest period of data available (60 years), which helps to balance out data variations in the interim censuses as much as possible. Finding a, b, and c by substituting these three data values into the above with n = 30 yields a logistic function.

$$P_T = \frac{13.37}{1 + 4.29e^{-0.025(T - 1950)}} \quad (11)$$

Since the growth is asymptotic to this amount, the maximum theoretical population of 13.37 billion is never actually reached in reality. Similarly, a normal distribution function is obtained by utilizing the same data to determine p, q, and r.

$$P_T = 2.53e^{0.0209(T - 1950) - 0.00007(T - 1950)^2}. \quad (12)$$

The highest population reached, using (9) and (10) respectively, is 12.07 billion in 2099. Figure 5 displays the global population based on the provided data extrapolated back to 1800 and forward to 2300, a duration of 500 years. The population can be found for any year from (11) and (12).

As long as inequality (5) is met, this method could theoretically take into account the population of every given nation or region rather than the worldwide total. Due to their continued exponential population growth, several nations—especially those in sub-Saharan Africa—do not meet the criteria for inequality. However, other nations in Eastern Europe, whose populations have decreased in recent decades, likewise do not fit the logistic model; instead, they follow the normal model.

CONCLUSION

The equations used here merely represent potential projections without providing any degree of likelihood, and population projection is rife with uncertainty. In contrast to the above instances, where a maximum figure of 12 to 14 billion is forecast from the supplied statistics, the UN Population Division predicts a leveling off, if not mild decline, of the world population after it hits between 9 and 9½ billion by 2050. However, because to the significant degree of uncertainty in UN projections, all population projections are given three alternative variations: low, medium, and high.

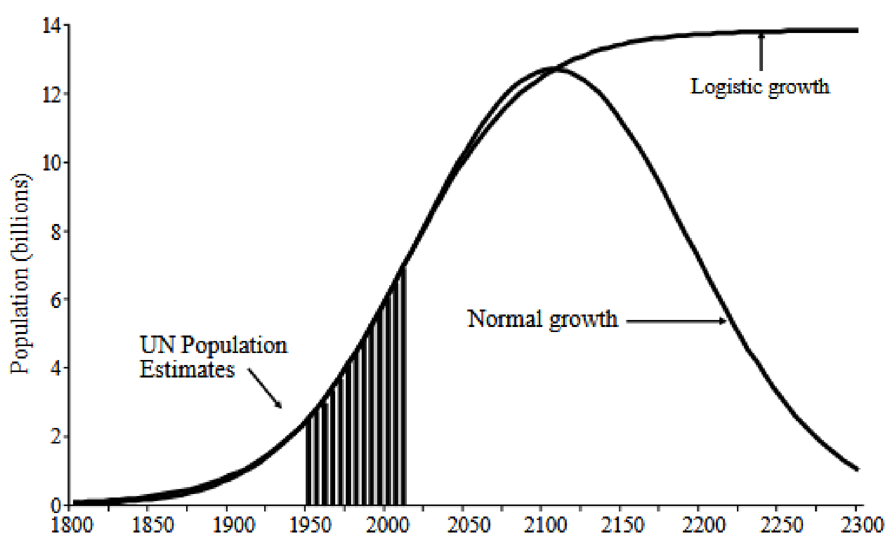

Figure 5: World Population (Billions) From 1800 With Projections To 2300 Based On 1950, 1980 And 2010 UN Population Estimates

Population change cannot be explained solely by mathematical formulas because it is really a socially determined human behavior issue. However, one could argue that the extremely complicated process of population projection can be attempted for the first time using simple mathematics. The same data inputs into two distinct functions can produce radically different long-term forecasts; stability with the logistic function and decrease with the normal function, as this work has further demonstrated. The catastrophe scenario is produced by unchecked exponential growth, stability is provided by logistic growth, and the final extinction scenario—an unpleasant but not impossible situation, as Easter Island has demonstrated—is produced by normal function growth.

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